Formula sheet

Numbers: $z = x + iy$ (algebraic form), $x, y \in \mathbb{R}$, $i^2 = -1$, $\overline{z} = x - iy$

Real and imaginary parts: $x = \mathrm{Re}(z) = \dfrac{z + \overline{z}}{2}$, $y = \mathrm{Im}(z) = \dfrac{z - \overline{z}}{2i}$

Basic operations: If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$, $z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2)$

$$\frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}, \quad z_2 \neq 0$$

Polar form: $z = r(\cos\theta + i\sin\theta)$, $r \geq 0$, $\theta \in (-\pi, \pi]$

Modulus: $r = |z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$

Argument: $\theta = \mathrm{Arg}(z)$ (principal value), $\arg(z) = \mathrm{Arg}(z) + 2\pi k$, $k \in \mathbb{Z}$

Identities: $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$, $|\overline{z}| = |z|$, $\arg(\overline{z}) = -\arg(z)$

$|z_1z_2| = |z_1||z_2|$, $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$

$\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Triangle inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$

De Moivre's theorem: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, $n \in \mathbb{Z}$

Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

Exponential form: $z = re^{i\theta}$

Functions: $w = f(z) = f(x + iy) = u(x, y) + iv(x, y)$

Complex exponential: $e^z := 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$

$e^{z_1+z_2} = e^{z_1}e^{z_2}$, $e^z = e^{x+iy} = e^xe^{iy} = e^x(\cos y + i\sin y)$

Trigonometric functions: $\cos z := \dfrac{e^{iz} + e^{-iz}}{2}$, $\sin z := \dfrac{e^{iz} - e^{-iz}}{2i}$

Complex Logarithm: $\mathrm{Log}\, z := \mathrm{Log}\,|z| + i\,\mathrm{Arg}\, z$ (principal value)

Derivatives: $f'(z_0) = \displaystyle\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$

Cauchy-Riemann equations: $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

Laplace's equation: $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = 0$, solutions are called *harmonic*

Integrals: $\int_\gamma f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt$, where γ is a simple smooth curve parameterized by $z(t)$, $a \leq t \leq b$.

Cauchy's Integral Theorem: $\oint_\Gamma f(z)\,dz = 0$.

Cauchy's Integral Formula: $f(z_0) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{z - z_0}\,dz$.

Cauchy's Differentiation Formula: $f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz$.

Series: Geometric series: $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$ if $|z| < 1$

Taylor series: $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$

Laurent series: $f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k$ with $a_k = \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{(z-z_0)^{k+1}}\,dz$

Singularities: *Removable* if $a_n = 0$ for all $n < 0$.
Pole of order m if $a_{-m} \neq 0$, but $a_n = 0$ for all $n < -m$.
Essential if $a_n \neq 0$ for infinitely many $n < 0$.

Residues: $\mathrm{Res}(f, z_0) = a_{-1}$

If $f(z) = h(z)/g(z)$ has a single pole at z_0, then $\mathrm{Res}(f, z_0) = h(z_0)/g'(z_0)$

For a pole of order m: $\mathrm{Res}(f, z_0) = \lim\limits_{z \to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)]$

Residue Theorem: $\oint_\Gamma f(z)\,dz = 2\pi i \sum_k \mathrm{Res}(f, z_k)$ for poles z_k inside Γ

Additional results: If $f(z) = P(z)/Q(z)$ with polynomials P, Q such that $\deg Q \geq 2 + \deg P$, then $\lim\limits_{R\to\infty} \int_{C_R^+} f(z)\,dz = 0$ for $C_R^+ = \{z \in \mathbb{C} \mid |z| = R,\ \mathrm{Im}(z) \geq 0\}$.

Jordan's Lemma: If $m > 0$ and $f(z) = P(z)/Q(z)$ with polynomials P, Q such that $\deg Q \geq 1 + \deg P$, then $\lim\limits_{R\to\infty} \int_{C_R^+} f(z)e^{imz}\,dz = 0$

Liouville's Theorem: If f is bounded and entire, then f is constant.

Maximum modulus principle: If f is analytic in a domain D and $|f(z)|$ achieves its maximum value at a point z_0 in D, then f is constant in D.

Argument Principle: If f is analytic and nonzero at each point of a simple closed positively-oriented contour C and is meromorphic inside C, then $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = Z - P$ where Z and P are, respectively, the number of zeros and poles of f inside C (counting multiplicities).

Rouché's Theorem. If f and h are analytic inside and on a simple closed contour C and $|h(z)| < |f(z)|$ on C, then f and $f + h$ have the same number of zeros (counting multiplicities) inside C.